Reassessing S4 Applied Optimization

S4: I can use derivatives to solve applied optimization problems.

Where topic was first introduced: Module 11

See also the reassessment guide for D8 for optimization details.

Video: This video is a 37 minute lecture covering a summary of optimization followed by an applied example. (`https://youtu.be/Q05jaW2L3ck`)

Basic Principles

- Extreme values can only happen at end points and places where the function might change direction (switch from increase to decrease or vice versa.)
- *Critical point* - We say a function, f, has a *critical point* at $x = p$ if p is **in the domain** of f and $f'(p) = 0$ OR $f'(p)$ is undefined. The point $(p, f(p)$ on the graph of f is also called a *critical point.*
  ("in the domain" means that the parent function exists at that point, or that $f(p)$ exists.)
  There are three basic parts to check: domain, $f' = 0$, and f' DNE. You must check for BOTH where f' is 0 or undefined, OR STATE why there are no x values where f' is 0 or undefined. Otherwise once you start testing your results could be flawed.
- Functions **might** change direction when the derivative is neither positive nor negative, so when $f'(x) = 0$ or $f'(x)$ DNE. Between any two such consecutive x values, the function CANNOT change direction because if it did there would have been another place where $f'(x) = 0$ or $f'(x)$ DNE. Thus if the function is increasing at ONE x value between two possible directions changes, the function is increasing for ALL values on that interval.
- Start by modeling the situation with a function of a single variable. Use a constraint function if the objective function has more than one input variable.
- Identify the endpoints and critical points.
- Test these values to conclude if there are extreme values because not all critical points correspond to a direction change!
- Tests that you can use:
  - First Derivative Test: use intervals of increase or decrease to determine if a function switches from increasing to decreasing (local max) or decreasing to increasing (local min) or neither at critical points. Can also be used to classify endpoints: for example, if a function decreases to the right of the left endpoint, then the left end point must have been higher than nearby values and thus must be a local max.
  - Second derivative test: IF your critical point corresponds to a horizontal tangent line (zero slope) and IF f'' is not zero, then you can use concavity to determine extrema: Let cp be a critical point. If $f'(cp) = 0$ and $f''(cp) > 0$, then there is a local minimum when $x = cp$. If $f'(cp) = 0$ and $f''(cp) < 0$, then there is a local maximum when $x = cp$. Otherwise this test is inconclusive or doesn't help us.

– Test for Global Extrema: *If* your function is continuous on a closed interval and *if* you only care about global extrema, then you can first verify (and state/show) that f is continuous on the closed interval and then compare all the output values of critical points and end points. The largest output is the global max and the smallest output is the global min. You MUST state that you are using this test and that you can use it by showing the "continuous on a closed interval" piece.

- Testing for extrema and working with parameters, which adds a layer to the challenge of optimization. There are some strategies that can make this easier:
  1. If you end up trying to take the square root of a negative number while solving for a critical point, this tells you that there is no critical point there.
  2. Consider rewriting negative exponents as fractions: for example my mind might look at $x^{-1/2}$ and think, oh, this exists if $x = 0$, but if I rewrite as $\frac{1}{x^{1/2}}$ it's painfully obvious that if $x = 0$ the fraction does NOT exist (can't divide by 0).
  3. Just because a place where f' DNE is NOT a critical point doesn't mean that something interesting isn't happening there. Test between all places where f' is 0 or undefined.
  4. The second derivative test can be easier with parameters.
  5. Sometimes you can't use the second derivative test so you need the first derivative test, which means you need to find x values larger or smaller than your critical point. For example, if your critical point is $a/2$, a number smaller than $a/2$ is $\frac{a/2}{2} = a/4$, and a number bigger than $a/2$ is $a/2 \cdot 2 = a$. (Basically I cut the critical point in half to get a smaller number, and I doubled it to get a larger number.)

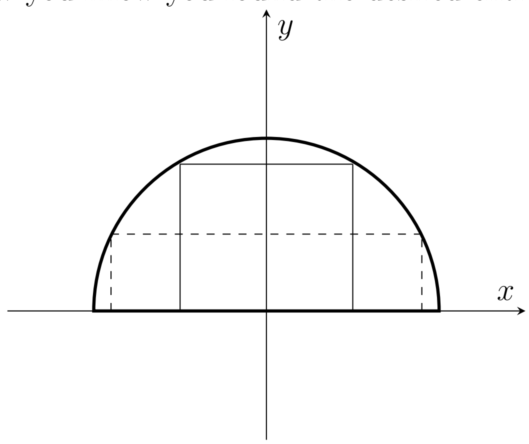
Favorite Mistakes:

- Not checking for where the derivative could be undefined. Even if the derivative is always defined, say this to let us know you checked! Otherwise we don't know you have found all possible direction changes and your First Derivative Test and The Test for Global Extrema fall apart.
- Not TESTING the critical points!! Don't just assume that your critical point is a max or min, test it and show us this is the case! Use one of the three tests listed above.
- Saying the First Derivative test is setting $f'(x) = 0$ and asking where $f'(x)$ DNE.
- Using the Second derivative test by looking at concavity on either side of the critical point. (Can you create a counterexample to see why this is wrong?)
- Saying that $f''(cp) = 0$ means there is neither a local max nor a local min at the critical point. (The Second Derivative Test is INCONCLUSIVE if you get 0 as a result. In other words, the test fails to tell you anything at all. Play around with $f(x) = x^4$ to see if this makes sense.)

Examples:

1. What are the dimensions of the largest rectangle which can be inscribed in the region bounded by $\sqrt{4-x^2}$ and by the x-axis. Assume that the rectangle is centered. Be sure to use calculus to justify where you search for extrema and how you know you found the desired extreme value.

   Sketch a picture!!! Graph $\sqrt{4-x^2}$ in Desmos, then add that to your paper. Sketch a few possible rectangles that would fit with their upper corners touching the semicircle, and the base of the rectangles are on the x-axis. The height of the rectangles correspond with the y-coordinate where the corners touch the semicircle, so the height $= y$. The base of the rectangles runs between the two corners, so from $-x$ to x, so the width is $x - -x$ or $2x$. Thus Area is $A = 2xy$.

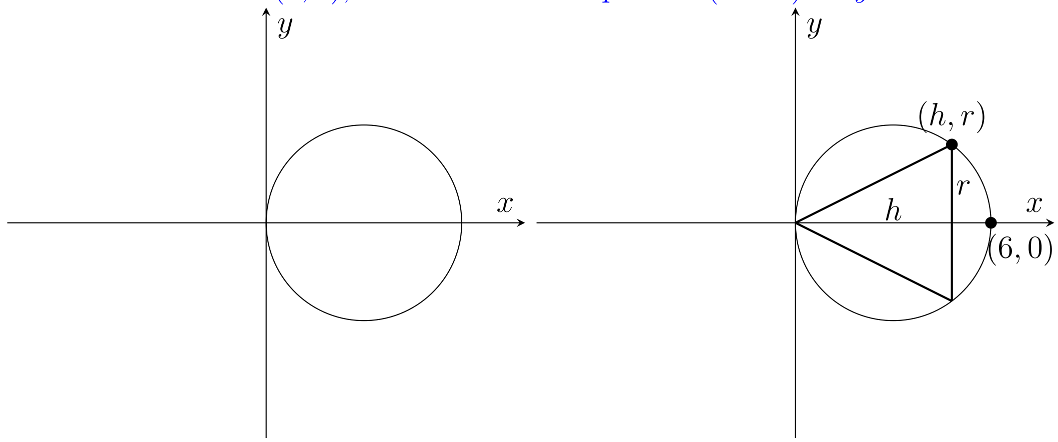
   

   Since $y = \sqrt{4-x^2}$, we have $A(x) = 2x\sqrt{4-x^2}$.

$$\begin{aligned}
A(x) &= 2x(4-x^2)^{1/2} \\
A'(x) &= 2\sqrt{4-x^2} + 2x(1/2)(4-x^2)^{-1/2}(2x) \\
&= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} \\
&= 2\sqrt{4-x^2} \cdot \frac{\sqrt{4-x^2}}{\sqrt{4-x^2}} - \frac{2x^2}{\sqrt{4-x^2}} \\
&= \frac{2(4-x^2)}{\sqrt{4-x^2}} - \frac{2x^2}{\sqrt{4-x^2}} \\
&= \frac{2(4-x^2) - 2x^2}{\sqrt{4-x^2}} \\
&= \frac{8 - 2x^2 - 2x^2}{\sqrt{4-x^2}} \\
&= \frac{8 - 4x^2}{\sqrt{4-x^2}}
\end{aligned}$$

   $A'(x) = 0$ if $8 - 4x^2 = 0 \implies 4(2-x^2) = 0 \implies 2 = x^2$ so when $x = \pm\sqrt{2}$.
   $A'(x)$ is undefined if $\sqrt{4-x^2} = 0 \implies (\sqrt{4-x^2})^2 = 0^2 \implies 4 - x^2 = 0 \implies 4 = x^2$ or if $x = \pm 2$.
   We can think of the domain as $0 \le x \le 2$ and then our only critical points are $x = \sqrt{2}, 2$.
   Since $x = 0$ and $x = 2$ are end points and yield 0 area (Try it- plug 0 and 2 into the area function above!), and since $A'(1) > 0$ and $A'(1.5) < 0$ therefore the maximum is when $x = \sqrt{2}$, and then $y = \sqrt{4 - (\sqrt{2})^2} = \sqrt{2}$.

2. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.
Orientation is really helpful in this problem. Let's think about it in 2-D and cut the sphere in half so we can see the dimensions of the cone. The center of the sphere will look like a circle. Let's center the circle at $(3, 0)$, so we'll use the equation $(x - 3)^2 + y^2 = 3^2$. Next we'll add the cone.



Notice that by setting up the image this way, the height of our cone is just the x coordinate and the radius of the cone is the y coordinate.
Volume of a cone is $(4/3)\pi r^2 h$, and this is the objective function. Using the formula of the circle as the constraint, we have $(h-3)^2 + r^2 = 3^2$. Solve the constraint function for r^2 to easily substitute into the volume equation:
$r^2 = 9 - (h - 3)^2$ and then
$V = (4/3)\pi(9 - (h - 3)^2)h = (4/3)\pi(9 - (h^2 - 6h + 9))h = (4/3)\pi(-h^2 + 6h)h$
$=(4/3)\pi(-h^3 + 6h^2)$. Now we are ready to optimize. The endpoints are the diameter of the circle, so $0 \leq h \leq 6$.
$V'(h) = (4/3)\pi(-3h^2 + 12h)$
$V'(h)$ is always defined because this is a polynomial.
$V'(h) = 0$ if:

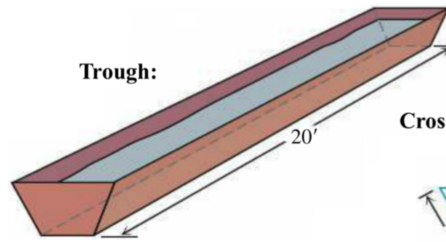
$$\begin{aligned} 0 &= (4/3)\pi(-3h^2 + 12h) \\ 0 &= (-3h^2 + 12h) \\ 0 &= h(-3h + 12) \end{aligned}$$

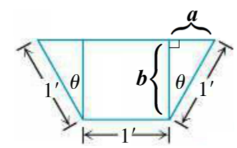
Either $h = 0$ or $(-3h + 12) = 0$ meaning our critical points are 0 and 4.
Now $V(h)$ is a continuous function on the closed interval $[0, 10]$ and
$V(0) = (4/3)\pi(-3 \cdot 0^2 + 12 \cdot 0) = 0$ and
$V(4) = (4/3)\pi(-3 \cdot 4^2 + 12 \cdot 4) \approx 134$ and
$V(10) = (4/3)\pi(-3 \cdot 10^2 + 12 \cdot 10) \approx -1675.5$. The largest of these output values is the global maximum by the Test for Global Extrema. Thus the largest volume is approximately 134.

Prepare for Revision:

1. What are the dimensions of the closed cylindrical can that has surface area 280 π square centimeters and contains the maximum volume?

2. The water trough in the figure is 20 feet long and 1 foot wide at the bottom. Only the angle θ can be varied. $a \geq 0$ and $b > 0$.





   (a) We want to maximize the volume of water the trough can hold. Find a function whose output represents the volume of the trough. It can be in terms of both a and b.

   (b) Now rewrite the function in terms of one variable. What happens if we rewrite b in terms of a?

   (c) What if we rewrite the volume of the trough in terms of the angle θ.

   (d) What is the domain for this function? (In terms of θ.)

   (e) What value of θ will maximize the trough's volume? Use calculus to make an analytic argument to justify your result. (This means explore ALL possibilities for critical points, and TEST the critical point(s) to verify that you have a maximum.) [Hint: You will want to use the Pythagorean Identity $\cos^2(\theta) = 1 - \sin^2(\theta)$.]